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Remarks on Kowalevski's top

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Abstract

We present a review of some results on Kowalevski's top (KT) in classical and quantum mechanics. The following items are considered:

- the generalized KT (GKT) on the Lie algebras $so(4)$, $e(3)$ and $so(3,1)$;
- Kowalevski's gyrostat on these algebras;
- action of the GKT;
- quantum counterparts of the KT;
- semiclassical quantization of the GKT;
- generalization of the KT by Chaplygin and Goryachev at $l = 0$ and its Lax representation.

Unsolved questions are also discussed.

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Introduction

In December 1888 the Institut of France awarded the Bordin prize to Sophie Kowalevski for her solution to one of the most sophisticated dynamical systems of the 19th century. The original Kowalevski top (KT) is defined on the orbits of the Euclidean Lie algebra $e(3)$ with generators $J_i, x_i,\ i = 1, 2, 3$, obeying the following Poisson brackets:

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k \qquad \{J_i, x_j\} = \varepsilon_{ijk} x_k \qquad \{x_i, x_j\} = 0. \tag{1}$$

The position of a rigid body is fixed by the components x_i of the Poisson vector which are cosines between the axis of the body frame and the field. J_i are the components of the angular momentum in the body frame.

The orbits are distinguished by fixing the values of the Casimir elements

$$l = \sum_i J_i x_i \qquad a^2 = \sum_i x_i x_i. \tag{2}$$

The general Hamiltonian of the top in a homogeneous field reads as a quadratic form in angular momentum and a linear form in the Poisson vector

$$H_{\text{general}} = \frac{1}{2} \sum_i \frac{J_i^2}{I_i} - \sum_i b_i x_i \tag{3}$$

where I_i is the principal momenta of inertia and b_i describe the interaction. Kowalevski proved that H_{general} generates integrable systems in three and only three special cases of the parameters involved, discovered the third case and integrated it in terms of ratios of hyperelliptic functions. The Hamiltonian of the KT top is

$$H = \tfrac{1}{2}(J_1^2 + J_2^2 + 2J_3^2) - bx_1. \tag{4}$$

It generates equations of motion by the standard rule

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cdot) = \{H, \cdot\}. \tag{5}$$

The additional constant of motion that Poisson commutes with H reads in terms of $J_\pm = J_1 \pm \mathrm{i}J_2$, $x_\pm = x_1 \pm \mathrm{i}x_2$ as

$$K = k_+k_- \qquad k_\pm = J_\pm^2 + 2bx_\pm \tag{6}$$

$$\{H, K\} = 0 \tag{7}$$

hence the KT is a completely integrable system by the Liouville theorem.

Kowalevski proved [1] that in new variables $w_{1,2}$ depending on values of the integrals of motion and the Casimir elements

$$w_{1,2} = \frac{R(J_+, J_-) \pm \sqrt{R(J_+, J_+)R(J_-, J_-)}}{(J_+ - J_-)^2} \tag{8}$$

$$R(J_+, J_-) = -J_+^2 J_-^2 + 4E J_+ J_- + 4bl(J_+ + J_-) + 4b^2a^2 - K \tag{9}$$

the equations of motion read

$$\varepsilon_i \dot{w}_i(w_1 - w_2) = \sqrt{-2R_5(w_i)} \qquad \varepsilon_i = (-1)^{i+1} \quad i = 1, 2 \tag{10}$$

$$R_5(w) = (w^2 - K)((w + 2E)(w^2 + 4b^2a^2 - K) - 8b^2l^2). \tag{11}$$

This equation shows that the motion is related to the Kowalevski algebraic curve

$$\Gamma_K: \quad v^2 = -2R_5(w). \tag{12}$$

Kowalevski was able to express all of the dynamical variables in terms of w_i and to integrate (10) as Θ functions. Her solution gave a strong impetus to the theory of integrable systems and gave rise to an extended literature on the subject.

Soon after the original paper [1] further studies concentrated on the proof of the Kowalevski theorem claiming that there are only three integrable cases for the Hamiltonian (3), clarifying the solution in Θ functions and further developing the idea of solutions of dynamical equations in the complex t-plane. A few generalizations of the KT top and some similar systems were found and some special cases were analysed. Mapping the KT to a natural dynamical system with two degrees of freedom admitting separation of variables in elliptic coordinates was constructed by Kolosov [2].

In the first years of quantum mechanics the quantum counterpart for the Kowalevski integral K was constructed by Lapporte [3].

Progress in the theory of integrable systems over the last 30 years has revived interest in the Kowalevski top. A few Lax representations were considered and analysed [4–6, 10–12], and the geometry and integration of the equations of motion of the top were studied in detail [8, 9, 13–15].

The aim of this paper is to outline some features of the present studies related mainly to the Kowalevski top in quantum mechanics.

1. The generalized KT (GKT) and Kowalevski's gyrostat on Lie algebras $so(4)$, $e(3)$ and $so(3,1)$

KT systems can be generalized in a few directions. In this section we consider substitution of the $e(3)$ algebra by two other rank-two Lie algebras [16, 25] and including in the Hamiltonian the additional linear in J_3 term [23, 24].

Suppose that local coordinates on the phase manifold instead of (1) form the following commutation relations of an algebra $\mathcal{E}$ depending on the constant $\mathcal{P}$:

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k \qquad \{J_i, x_j\} = \varepsilon_{ijk} x_k \qquad \{x_i, x_j\} = \mathcal{P}\varepsilon_{ijk} J_k \tag{13}$$

with the Casimir elements

$$l = \sum_i J_i x_i \qquad \tilde{a} = \sum_i (x_i x_i - \mathcal{P} J_i J_i). \tag{14}$$

We consider as special the algebra $\mathcal{E}$ at three special values of the parameter $\mathcal{P}$

$$\mathcal{E} = \begin{cases} so(4) & \text{for} \quad \mathcal{P} = 1 \\ e(3) & \text{for} \quad \mathcal{P} = 0 \\ so(3,1) & \text{for} \quad \mathcal{P} = -1 \end{cases} \tag{15}$$

so $\mathcal{E}$ is the Lie algebra of the symmetry group of the Kepler problem at positive, zero and negative energies, respectively.

The Hamiltonian on the orbits of the $\mathcal{E}$ algebra is taken again in the form (4)

$$H = \tfrac{1}{2}(J_1^2 + J_2^2 + 2J_3^2) - bx_1. \tag{16}$$

A slight modification of $k_\pm \to k_{\mathcal{E}\pm}$

$$k_{\mathcal{E}\pm} = J_\pm^2 + 2bx_\pm - \mathcal{P}b^2 \tag{17}$$

allows us to write the Kowalevski integrals on the orbits of the algebra $\mathcal{E}$ as

$$K_{\mathcal{E}} = k_{\mathcal{E}+} k_{\mathcal{E}-} \qquad \{H, K_{\mathcal{E}}\} = 0. \tag{18}$$

Another generalization of the KT is the Kowalevski gyrostat (KG) [23, 24]. This generalization also respects the KT on the algebra $\mathcal{E}$ [24].

The modified Hamiltonian of the gyrostat differs from that of the KT by a linear term,

$$H_{\mathrm{KG}} = \tfrac{1}{2}(J_1^2 + J_2^2 + 2(J_3 - \lambda)^2) - bx_1 \tag{19}$$

with λ being an arbitrary real parameter. The second integral of motion takes the form

$$K_{\mathrm{KG}} = k_{\mathcal{E}+} k_{\mathcal{E}-} + 4\lambda G_\lambda \tag{20}$$

$$G_\lambda = 2(J_3 - 2\lambda)(-\mathcal{P}b + J_+ J_-) + 2(J_+ + J_-)bx_3. \tag{21}$$

For the Euclidean algebra $e(3)$, i.e. at $\mathcal{P} = 0$, the additional term G_λ has already appeared in the theory of integrable tops. Namely, at the special value of the Casimir element $l = 0$, G_λ commutes with the Hamiltonian

$$H_{\mathrm{GCG}} = \tfrac{1}{2}(J_1^2 + J_2^2 + 4(J_3 - \lambda)^2) - bx_1 \tag{22}$$

that was originally found in 1900 by Goryachev and Chaplygin. Its quantum variant was studied in detail in [17]. Separation of variables was done, the spectrum of integrals of motion was calculated numerically and its asymptotics at weak and strong fields obtained. Analytical continuation of the eigenvalues to the complex values of coupling constants was analysed [18].

The Goryachev–Chaplygin top was incorporated into the Lax scheme. In classical mechanics the system possesses the 3×3 Lax matrix [20] which is one of the minors of the Reyman–Semenov-Tian-Shansky 4×4 Lax matrix for the Kowalevski gyrostat [10]. In addition to (21) it gives another indication of the close relations between Kowalevski and Goryachev–Chaplygin systems.

Another 2×2 Lax representation for the GCG which works both in classical and quantum mechanics was constructed by Sklyanin [19]. It satisfies the quadratic rational R-matrix algebra.

2. Action of the GKT and separation of variables

The action variables for the KT at first sight are an immediate consequence of the Kolosov mapping [2] of the KT to a two-degrees-of-freedom natural dynamical system admitting separation in elliptic coordinates. However, this mapping includes a change of time and therefore provides a non-canonical transform in four-dimensional phase space. The correct complex angle–action variables for the KT were originally constructed by Veselov and Novikov [21] (see also [22] and references therein). The KT action was a starting point for semiclassical quantization in [25].

It was observed in [21] that introduction of the new scaled variables (sometimes also used by Kowalevski [1]) instead of w_i

$$s_i = w_i + 2H \tag{23}$$

is preferable, because coordinates s_i are Poisson commuting

$$\{s_1, s_2\} = 0 \tag{24}$$

unlike those of w_i [1]. We rewrite the equations of motion in s_i variables as

$$\varepsilon_i \dot{s}_i (s_1 - s_2) = \sqrt{-2R_5(s_i)} \qquad \varepsilon_i = (-1)^{i+1} \quad i = 1, 2 \tag{25}$$

with a new definition of the polynomial $R_5(s)$

$$\begin{aligned} R_5(s) &= (s^2 - 2Hs + \kappa - 2b^2a^2)(s(s^2 - 4Hs + \kappa + 2b^2) - 8b^2l^2) \\ \kappa &= 4H^2 - K + 2b^2a^2. \end{aligned} \tag{26}$$

Equations (25) completely define the motion of the system. All other dynamical variables can be expressed in terms of s_i and values of E, l, K, a^2. Because s_i Poisson commute, one can hope to restore the conjugated momenta and then the stationary action takes a form which is typical for the separation of variables, i.e. a sum of two summands $S = S_1(s_1, E, K, b, l) + S_2(s_2, E, K, b, l)$ each depending on values of integrals of motion and one dynamical variable (23) only. As was shown in [26] momenta p_i canonically conjugated to the coordinates s_i are defined by quadratures

$$p_i = 2 \int \frac{\partial H(s, \dot{s})}{\partial \dot{s}_i} \frac{\mathrm{d}\dot{s}_i}{\dot{s}_i} \tag{27}$$

where H is written with the help of (25) via $s_{1,2}, \dot{s}_{1,2}$ at the level of constants of motion.

[1] The detailed analytical computation of Poisson brackets $\{w_1, w_2\} \neq 0$ and $\{s_1, s_2\} = 0$ is not published. I did this with Maple V by substituting in the definition of the coordinates (8) and (23) the integrals of motion and the Casimir elements as functions of dynamical variables.

Taking the quadratures and excluding the velocities with the help of (25) we arrive at one-dimensional separated equations depending on common values of integrals of motion and the Casimir elements

$$s_i^2 - 4Hs_1 - \frac{4b^2l^2}{s_i} + \kappa = 2b^2\left(a^2 - \frac{2l^2}{s_i}\right)\cos\left(2\sqrt{2s_i}\,p_i\right) \qquad i = 1, 2. \tag{28}$$

These equations can be written in terms of a direct sum of generators of two auxiliary rank-one Lie algebras. Minding (24) one proves that quantities

$$\lambda_i^{\pm} = \exp\left(\pm 2\sqrt{-2s_i}\,p_i\right) \tag{29}$$

and $u_i = \sqrt{s_i}$ obey the Lie algebra $e(2) \oplus e(2)$ relations with the Poisson brackets

$$\{u_i, \lambda_k^{\pm}\} = \mp\mathrm{i}\sqrt{2}\lambda_k^{\pm}\delta_{ik} \qquad \{\lambda_i^+, \lambda_k^-\} = 0 \qquad \{u_i, u_k\} = 0 \qquad i, k = 1, 2 \tag{30}$$

where the Casimir element is $\lambda^+\lambda^- = 1$ in both cases. The one-dimensional equations now read

$$u_i^6 - 4Hu_1^4 + \kappa u_i^2 - 4b^2l^2 = b^2(a^2u_i^2 - 2l^2)(\lambda_i^+ + \lambda_i^-) \qquad i, k = 1, 2 \tag{31}$$

or in brief notation

$$P_3(u^2) = P_1(u^2)(\lambda_i^+ + \lambda_i^-) \qquad i, k = 1, 2 \tag{32}$$

$$P_3(u^2) = u^6 - 4Hu^4 + \kappa u^2 - 4b^2l^2 \qquad P_1(u^2) = b^2(a^2u^2 - 2l^2). \tag{33}$$

In view of the definitions of $\lambda^{\pm}$ (29) these equations can be treated as a new algebraic curve $\Gamma_{\text{separated}}$ connecting u and $v = b^2(a^2u^2 - 2l^2)\lambda^+$

$$\Gamma_{\text{separated}}\colon \quad v^2 - P_3(u^2)v + P_1(u^2)^2 = 0. \tag{34}$$

Reyman and Semenov-Tian-Shansky [10] constructed the Lax pair describing a three-degrees-of-freedom generalization of the Kowalevsi gyrostat to the case of two homogeneous fields. For their system the motion is related to the curve

$$\Gamma_{\text{R–STS}}\colon \quad \mu^4 - 2d_1(\lambda^2)\mu^2 + d_2(\lambda^2) = 0 \tag{35}$$

where $d_1(z)$ and $d_2(z)$ are proportional to the polynomials $\mathcal{P}_2(z)$ and $\mathcal{P}_3(z)$ of the second and the third order, respectively, with coefficients being constants of motion and a gyrostat parameter

$$d_1(z) = \frac{1}{z}\mathcal{P}_2(z) \qquad d_2(z) = \frac{1}{z^2}\mathcal{P}_3(z). \tag{36}$$

Recently, it was shown [27] how to pass from the Kowalevski (12) curve to that of Reyman and Semenov-Tian-Shansky (35) at $l = 0$ by the Richelot transform. At arbitrary values of the Casimir operator the l relation of these curves has not yet been studied.

For $\mathcal{E}$ algebra the polynomials (33) entering the curve (34) are modified to

$$P_{3\mathcal{E}}(u^2) = u^6 - (4H + \mathcal{P}b^2)u^4 + \kappa_{\mathcal{E}}u^2 - 4b^2l^2 \tag{37}$$

$$\begin{aligned} &P_1(u^2) \to P_{2\mathcal{E}}(u^2) = b^2(\mathcal{P}u^4/2 + a^2u^2 - 2l^2) \\ &\kappa_{\mathcal{E}} = (2H + \mathcal{P}b^2)^2 - K + 2b^2\tilde{a} \end{aligned} \tag{38}$$

so for $so(4)$ and $so(3, 1)$ algebras the polynomial $P_1(u^2)$ changes its degree from one to two. The corresponding curve of the separated equations now reads

$$\Gamma_{\mathcal{E}\,\text{separated}}\colon \quad v^2 - P_{3\mathcal{E}}(u^2)v + P_{2\mathcal{E}}(u^2)^2 = 0. \tag{39}$$

The structure of this curve can be helpful in the search for an extension of the Reyman and Semenov-Tian-Shansky Lax matrix [10] to $so(4)$ and $so(3, 1)$ Lie algebras.

3. Quantum counterparts of the KT

In quantum mechanics the Kowalevski top was originally considered by Lapporte [3] who wrote down the integrals of motion. In [16, 24] the quantum integrals of motion were extended to Lie algebras $so(4)$ and $so(3, 1)$ both for the Kowalevski top and gyrostat.

As a quantum analogue of (13) we take

$$[J_i, J_j] = \mathrm{i}\hbar\varepsilon_{ijk}J_k \qquad [J_i, x_j] = \mathrm{i}\hbar\varepsilon_{ijk}x_k \qquad [x_i, x_j] = \mathrm{i}\hbar\mathcal{P}\varepsilon_{ijk}J_k \quad (40)$$

where for $\mathcal{P} = -1, 0, 1$ according to (15) we have three algebras $o(4)$, $e(3)$ and $o(3, 1)$, with the Casimir elements

$$l = \sum_i J_i x_i \qquad \tilde{a} = \sum_i (x_i x_i - \mathcal{P} J_i J_i). \quad (41)$$

The quantum counterparts of the generalized Kowalevski top (GKT) and gyrostat (GKG) on the three algebras are defined by the Hamiltonian

$$H_{\mathrm{GKT}} = \tfrac{1}{2}(J_1^2 + J_2^2 + 2J_3^2) - bx_1 \quad (42)$$

$$H_{\mathrm{GKG}} = \tfrac{1}{2}(J_1^2 + J_2^2 + 2(J_3 - \lambda)^2) - bx_1. \quad (43)$$

Introducing again (17)

$$k_{\mathcal{E}\pm} = J_\pm^2 + 2bx_\pm - \mathcal{P}b^2 \quad (44)$$

we can check the following set of commutators:

$$[k_{\mathcal{E}+}, J_3] = -\hbar(k_{\mathcal{E}+} + J_+^2 + \mathcal{P}/4) \quad (45)$$

$$[k_{\mathcal{E}+}, J_-] = 2[H_{\mathrm{GKT}}, J_+] \quad (46)$$

$$[k_{\mathcal{E}+}, H_{\mathrm{GKT}}] = -2\hbar\{J_3, k_{\mathcal{E}+}\}_+ \quad (47)$$

where $\{\, , \}_+$ denotes an anticommutator. From these relations follow commutation relations with the Hamiltonian

$$\left[H_{\mathrm{GKT}}, \tfrac{1}{2}\{k_{\mathcal{E}+}, k_{\mathcal{E}-}\}\right] = -2\hbar^2[H_{\mathrm{GKT}}, \{J_+, J_-\}_+] = -\hbar\tfrac{1}{2}[J_3, [k_{\mathcal{E}+}, k_{\mathcal{E}-}]] \quad (48)$$

$$[H_{\mathrm{GKT}}, [k_{\mathcal{E}+}, k_{\mathcal{E}-}]] = -\hbar[J_3, \{k_{\mathcal{E}+}, k_{\mathcal{E}-}\}] \quad (49)$$

connecting commutators with $[H_{\mathrm{GKT}}, \cdot]$ and $[J_3, \cdot]$. These allow us to verify that the operator

$$K_{\mathrm{GKG}} = \tfrac{1}{2}\{k_{\mathcal{E}+}, k_{\mathcal{E}-}\} + \frac{\lambda}{\hbar}[k_{\mathcal{E}+}, k_{\mathcal{E}-}] + (2\hbar^2 - 8\lambda^2)\{J_+, J_-\}_+ \quad (50)$$

commutes with H_{GKG} on the Lie algebra (40).

To construct quantum counterparts of the Kowalevski top one can also start from a quantum variant of the integral K instead of the Hamiltonian. Then one can take Hermitian operators

$$K_+ = k_{\mathcal{E}+}k_{\mathcal{E}-} \qquad \text{or} \qquad K_- = k_{\mathcal{E}-}k_{\mathcal{E}+} \quad (51)$$

alternatively to be new quantum constants of motion. These operators generate an integrable system with the Hamiltonians

$$H_{\mathrm{GKT}\pm} = H_{\mathrm{GKT}} \pm 4\hbar J_3 \quad (52)$$

i.e. quantum corrections transform the Kowalevski top into the Kowalevski gyrostat (see (19)) with the gyrostatic parameter being $\pm 2\hbar$. This observation gives a special status to the gyrostat generalizations of the top.

In further studies of the quantum KT it is natural to use the known Lax matrices. Actually until now there has been no activity along this line.

Lax matrices by Haine and Horozov [5] and Adler and van Moerbeke [6, 7] use a transition to another algebra, i.e. a non-canonical transform. The Haine–Horozov result admits generalization to $so(4)$, $so(3, 1)$ algebras [25]. This family of Lax matrices appeared in the geometrical analysis of the original Kowalevski derivation. The dependence of the Kowalevski separated coordinates on the values of integrals of motion may be an indication of the application of the Euler–Maupertuis ideas. Actually until now the gyrostatic term has not been included in these Lax matrices and it is another obstacle to using these results in the quantum case.

The Lax matrix by Reyman and Semenov-Tian-Shansky [10] satisfies the linear r-matrix algebra with a rational r-matrix. Hence it easily admits a quantum treatment. However, the dimension of auxiliary space in their case is four or five and it is also unclear how to extend the Lax matrix [10] to the $so(4)$, $so(3, 1)$ algebras. Until now nobody had applied this Lax matrix to the study of the spectrum of constants of motion.

4. Semiclassical quantization of the GKT [25]

Taking the quadratures for momentum (27) and excluding $\dot{s}_i$ with the help of equations of motion (25) one obtains the momenta as functions of integrals of motion and coordinates s_i

$$\begin{aligned} p(s) &= \frac{1}{2\sqrt{-2s}} \ln \frac{x + \sqrt{x^2 + c^2}}{c} \qquad x = 2\sqrt{y^2 + cy} \\ y &= (s - H)^2 - K \qquad c = 4b^2\left(1 - \frac{2l^2}{s}\right). \end{aligned} \tag{53}$$

The action is a sum of two summands

$$S = S_1 + S_2 = \oint_{\alpha_1} p(s_1)\, \mathrm{d}s_1 + \oint_{\alpha_1} p(s_1)\, \mathrm{d}s_1. \tag{54}$$

Here α_i are a-cycles of the Jacobi variety of the algebraic curve $z^2 = -2R_5(s)$. The semiclassical spectrum follows from the system of nonlinear Bohr–Sommerfeld quantization conditions

$$S_i = 2\pi(n_1 + m_i/4) \qquad i = 1, 2 \qquad n_i = 0, 1, \ldots \tag{55}$$

where m_i are the Maslov indices.

Numerical data for semiclassical spectra of the integrals of motion were obtained by two different algorithms yielding integrals of motion as functions of the coupling constant b and quantum numbers. In the first approach we solved equations (55) numerically. The second approach used the property of the action integrals to be adiabatic invariant at slowly varying field strength and it is called the adiabatic switching method. As initial values the asymptotics of quantum integrals in weak and strong fields were used. The results appeared to be in good agreement which indirectly implies that the computations are correct.

5. Chaplygin and Goryachev generalization of the KT at $l = 0$. Its Lax representation in quantum mechanics

It was shown by Chaplygin in 1903 and Goryachev in 1916 [28] that at $l = 0$ the KT admits the inclusion of a few additional terms depending on x_1, x_2, x_3 in the KT Hamiltonian.

For this system Kuznetsov and Tsiganov [29] found a 2×2 Lax matrix which is related to the Lax matrix of a special Neumann's system with a Hamiltonian

$$H_{\mathrm{N}} = \tfrac{1}{2}(J_1^2 + J_2^2 - b^2 x_3^2) + \alpha/x_3^2 \tag{56}$$

both in classical and quantum mechanics. In the latter case, while Neumann's Lax matrix

$$L_{\mathrm{N}} = \begin{pmatrix} u^2 - 2J_3 u - J_1^2 - J_2^2 - \frac{1}{2} - 2\alpha/x_3^2 & \mathrm{i}b(x_+ u - \frac{1}{2}\{x_3, J_+\}) \\ \mathrm{i}b(x_- u - \frac{1}{2}\{x_3, J_-\}) & b^2 x_3^2 \end{pmatrix} \tag{57}$$

satisfies the Yang–Baxter-type quadratic algebra with the standard rational R-matrix ($R = u + \mathrm{i}\kappa P$, P is a permutation), the Kowalevski–Chaplygin–Goryachev top is connected with the reflection equation algebra

$$R(u-v)\overset{1}{\mathcal{T}}_-(u)R(u+v-\mathrm{i}\kappa)\overset{2}{\mathcal{T}}_-(v) = \overset{2}{\mathcal{T}}_-(v)R(u+v-\mathrm{i}\kappa)\overset{1}{\mathcal{T}}_-(u)R(u-v) \tag{58}$$

$$R(-u+v)\overset{1}{\mathcal{T}}{}_+^{t_1}(u)R(-u-v-\mathrm{i}\kappa)\overset{2}{\mathcal{T}}{}_+^{t_2}(v) = \overset{2}{\mathcal{T}}{}_+^{t_2}(v)R(-u-v-\mathrm{i}\kappa)\overset{1}{\mathcal{T}}{}_+^{t_1}(u)R(-u+v). \tag{59}$$

Here t_1 and t_2 denote matrix transpositions in the first and the second auxiliary space respectively, $\mathcal{T}_\pm(u)$ are

$$\mathcal{T}_-(u) = L_{\mathrm{N}}(u)K_-\big(u - \tfrac{1}{2}\mathrm{i}\kappa\big)\sigma_2 L_{\mathrm{N}}^T(-u)\sigma_2 \tag{60}$$

$$\mathcal{T}_+(u) = K_+\big(u + \tfrac{1}{2}\mathrm{i}\kappa\big) \tag{61}$$

and boundary condition c-number matrices read

$$K_-(u) = \begin{pmatrix} \alpha_1 & u \\ -\beta_1 u & \alpha_1 \end{pmatrix} \qquad K_+(u) = \begin{pmatrix} \alpha_2 & \beta_2 u \\ -u & \alpha_2 \end{pmatrix} \tag{62}$$

where α_i and β_i are arbitrary complex constants. The Lax matrix now is a product $L_{K\,Tsi}(u) = \mathcal{T}_+(u)\mathcal{T}_-(u)$ and its trace $\tau(u) = \mathrm{tr}\ L_{K\,Tsi}$ generates the constants of motion of the integrable system with the Hamiltonian

$$H_{\mathrm{KCGT}} = \tfrac{1}{2}(J_1^2 + J_2^2 + 2J_3^2) + c_1 x_1 + c_2 x_2 + c_3(x_1^2 - x_2^2) + c_4 x_1 x_2 + c_5/x_3^2 \tag{63}$$

where $c_1 = \frac{1}{2}\mathrm{i}b(\alpha_2 - \alpha_1)$, $c_2 = \frac{1}{2}b(\alpha_1 + \alpha_2)$, $c_3 = -\frac{1}{4}b^2(\beta_1 + \beta_2)$, $c_4 = \frac{1}{2}b^2\mathrm{i}(\beta_2 - \beta_1)$, $c_5 = \alpha$.

In the classical limit the characteristic polynomial of the Kuznetsov–Tsiganov Lax matrix for the Kowalevski top generates the curve $\det(L_{\mathrm{K\,Tsi}}(u) - v) = 0$, i.e.

$$\Gamma_{\mathrm{K\,Tsi}}\colon \quad v^2 - v\tilde{P}(u^2) + a^4 b^4 u^4 \tag{64}$$

$$\tilde{P}(u^2) = u^6 - 4Hu^4 + (K - 2b^2 a^2) \tag{65}$$

where a zero value of the spin Casimir of $e(3)$ $l = 0$ is already substituted. This curve should be compared with the other known curves for the Kowalevski top (34) and (35). It reminds one very much of the curve (34) associated with the separated equation.

Until now the separation of variables for (63) was not constructed except for the special case $c_1 = c_2 = c_4 = c_5 = 0$ considered by Chaplygin [28]. At the level $l = 0$ he wrote down the second integral of motion and its value as

$$Ch = (J_1^2 - J_2^2 + cx_3^2)^2 + 4J_1^2 J_2^2 = h^2 \tag{66}$$

where $c = 2c_3$. Then the separated coordinates

$$s_{1,2} = \frac{J_1^2 + J_2^2 \pm h}{cx_3^2} \tag{67}$$

allow us to parametrize the motion by the product of two elliptic curves.

In the quantum case the spectrum of the constants of motion generated by the Kuznetsov–Tsiganov Lax matrix is unknown.

The generalization (63) is valid only at a special value of the Casimir element $l = 0$. Tsiganov [30] developed an additive deformation scheme for R-matrix algebras allowing one to change $l = 0$ to an arbitrary value. The resulting algebra differs from the reflection equation algebra (58), (59) by linear terms. This scheme was applied [31] to search for a new 2×2 Lax matrix for the quantum Kowalevski top at arbitrary $l \neq 0$ starting from $L_{KTsi}(u) = \mathcal{T}_+(u)\mathcal{T}_-(u)$. Tsiganov [31] constructed a matrix with a trace reproducing the correct quantum integrals of motion. However, the determinant of his matrix contains not only Casimir elements, so it is unclear how to obtain the algebraic curve related to the motion in the Tsiganov approach. Now the scheme of [31] is incomplete and needs further analysis.

Final remark

I conclude with a (pessimistic or intriguing) statement that while in classical mechanics the Kowalevski top is studied rather extensively, for its quantum counterpart there are many open questions and even the spectrum of its integrals is still unknown.

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